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On waves generated in rotating stratified liquids by travelling forcing effects

By V. SUBBA RAO AND G. V. PRABHAKARA RAO
Department of Mathematics, Indian Institute of Technology, Madras-36, India

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The pattern and propagation of waves excited by forcing effects that may oscillate with a frequency σ_0 and are travelling with a uniform velocity U along the axis of rotation in a rotating stratified liquid are studied by employing the technique of Lighthill. The fluid is assumed to be an unbounded, inviscid, incompressible, non-diffusive, rotating stratified liquid, rotating with a constant angular velocity Ω about a vertical axis and with a density decreasing vertically upwards. It is found that the effect of rotation on stratification or vice versa is to reduce the region of disturbance and to split the wave crests for $\sigma_0 \neq 0$. The periodic nature of the forcing effect excites various systems of waves which are otherwise coincident.

1. Introduction

The phenomena occurring in a rotating stratified liquid differ markedly from that in a rotating liquid or in a stratified liquid [see, for example, Carrier 1965; Barcilon & Pedlosky 1967; Krishna & Sarma 1969, etc.]. Near thermoclines, the stratification effects are quite predominant and the effects of rotation are usually ignored. Similarly, in well-mixed regions away from thermoclines, only rotation is taken into account and not the stratification. But in a low-frequency movement of the ocean, or when the forcing effect has a low frequency, the motion seems to depend sufficiently on both rotation and stratification. Motivated by this, in this paper we have considered waves excited by forcing effects that may oscillate with a frequency σ_0 and travelling with a uniform velocity $(0, 0, U)$ in an inviscid, incompressible, rotating stratified liquid rotating with a uniform angular velocity Ω about a vertical axis.

The waves in a rotating sea with a variable density have been extensively treated by Love (1891), Fjedlstad (1933), Groen (1948) and Eckart (1960) and a brief sketch of this work is given in La Fonda (1962), Lighthill (1966) and Phillips (1966). When both rotation and stratification are taken into account, then the fluid is characterized by two natural frequencies, viz. 2Ω and N, where N is the Brunt–Väisälä frequency and their influence depends on whether N is greater or smaller than 2Ω. If $2\Omega < N$, then the rotation would modify the internal waves by adding a low-frequency cut-off at 2Ω to the existing high-frequency cut-off at N. When $N < 2\Omega$, which can also be of geophysical interest, the inertial modes are subjected to high- and low-frequency cut-offs at 2Ω and N respectively.

When a steady forcing effect translates with a uniform velocity U along the axis of a homogeneous rotating liquid, it excites two systems of waves (Lighthill 1967). The first system consists of waves of uniform length $\pi U/\Omega$ in all directions behind the forcing region with hemispherical crests. The second system consists of unattenuated waves propagating both ahead of and behind the forcing region in a 'Taylor column'. If we now introduce a density gradient (however small, it might be) such that $N < 2\Omega$ then it is shown in § 4 that the second system of waves propagating in the upstream direction is completely eliminated. All the downstream waves coalesce into a single system with cusp-shaped crests and propagate only downstream with a cone as their envelope. Here the waves independent of the vertical co-ordinate Z do not exist.

If the forcing region, instead of being steady, oscillates with a frequency σ_0 in a homogeneous rotating liquid then three cases will arise according as $\sigma_0 \lesseqgtr 2\Omega$ (see Nigam & Nigam 1962). For $\sigma_0 < 2\Omega$ there are three systems of waves, say $\mathscr{P}_1$, $\mathscr{P}_2$ and $\mathscr{P}_3$ with $\mathscr{P}_2$, $\mathscr{P}_3$ propagating only downstream each confined to a cone and $\mathscr{P}_1$ propagating in all directions except in an upstream cone. The crests of $\mathscr{P}_3$ are cusp-shaped and those of $\mathscr{P}_1$ and $\mathscr{P}_2$ extend to infinity. When $\sigma_0 \geqslant 2\Omega$, $\mathscr{P}_1$ disappears and $\mathscr{P}_2$, $\mathscr{P}_3$ propagate as in the above case, but for $\sigma_0 > 2\Omega$ the crests of $\mathscr{P}_2$ are also cusp-shaped. The crests of $\mathscr{P}_2$ and $\mathscr{P}_3$ always have a point in common, the point where they touch the axis of rotation, so that both the systems have a single wave front. When stratification is also present, one could broadly consider three cases according as $\sigma_0 \leqslant N$, $N < \sigma_0 < 2\Omega$ or $2\Omega \leqslant \sigma_0$. We will see in § 5 that the first and foremost effect of stratification for any $\sigma_0 \neq 0$, is to split the two systems $\mathscr{P}_2$ and $\mathscr{P}_3$ at their common point and to separate them. Here again, there are three systems of waves for $N < \sigma_0 < 2\Omega$ and two otherwise. For $\sigma_0 > N$ the propagation of waves is analogous to that in a homogeneous liquid with the exception that the shape of the crests is slightly modified and the enveloping conical regions are diminished. But the waves appearing for $\sigma_0 \leqslant N$ have no counterparts in the homogeneous case.

The wave pattern produced by a steady disturbance moving vertically in a stratified liquid has been investigated by Warren (1960), Mowbray & Rarity (1967*a*, *b*), Rarity (1967) and Lighthill (1967). Here the waves excited belong to a single system of waves propagating in all directions below the forcing effect with a 'flared skirt' shape of crests. In § 4, we will see that the effect of rotation on these waves is to eliminate longer waves whose directions are close to the horizontal and as a result the remaining waves get folded upon themselves and propagate downstream in a cone with cusp-shaped crests.

Axisymmetric internal waves generated by a travelling oscillating body have been studied both theoretically and experimentally by Stevenson (1969). We observe that for $0 < \sigma_0 < N$, the axisymmetric wave-number surface has two open infinite branches, one passing through the origin and the other through $(0, 0, -\sigma_0/U)$ and they are connected by a bulb-like closed portion. Let us denote them by $G_+(\sigma_0)$, $G_-(\sigma_0)$ and $G_1(\sigma_0)$ respectively. For frequencies in this range there are two system of waves, say $\mathscr{G}_1$ and $\mathscr{G}_2$. The systems $\mathscr{G}_1$, corresponding to $G_+(\sigma_0)$, propagates downstream confined to the cone

$$H_+(\sigma_0)\colon x^2+y^2-\cot^2\theta_1(z-Ut)^2 = 0, \quad z-Ut < 0,$$

where $\theta_1 = \tan^{-1}[\sigma_0/U(N^2-\sigma_0^2)^{\frac{1}{2}}]$. The second system $\mathscr{G}_2$, which is composed of waves corresponding to $G_-(\sigma_0)$ and $G_1(\sigma_0)$ propagates in all directions without penetrating into the extension of $H_+(\sigma_0)$ to the upstream side. For $\sigma_0 = 0$ the two systems coalesce into the single system mentioned in the above paragraph. The waves corresponding to $G_-(\sigma_0)$, for all σ_0, propagate close to the negative z-axis confined to the cone

$$H_-(\sigma_0)\colon x^2+y^2-\cot^2\theta_2(z-Ut)^2 = 0, \quad z-Ut < 0,$$

where $\theta_2 = \tan^{-1}(\sigma_0/UN)$, while those corresponding to $G_1(\sigma_0)$ propagate outside $H_-(\sigma_0)$. As $\sigma_0 \to N$, $\mathscr{G}_2$ spreads out and propagates in all directions while $\mathscr{G}_1$ shrinks and disappears completely when $\sigma_0 = N$. For $\sigma_0 > N$, $G_1(\sigma_0)$ becomes an infinite branch and has a monoclastic circle (a curve which divides the wave-number surface into synclastic ($\kappa > 0$) and anticlastic ($\kappa < 0$) regions. Hereafter we refer to this circle(s) by m.c.(s)). So the system $\mathscr{G}_2$ folds back on itself and propagates with kite-shaped crests in a cone $H_1(\sigma_0)$, the cone with the vertex at the forcing effect and with generators parallel to the normals on m.c. Stevenson (1969) also gives six schlieren photographs of the wave pattern generated by an oscillating sphere moving vertically in a stably stratified salt solution along with theoretically predicted shapes. The agreement between the theory and the experiments is excellent. If, besides stratification, the fluid is also subjected to rotation so that $2\Omega < N$ then we find that rotation splits the wave-number surface at $(0, 0, -\sigma_0/U)$, the point where $G_1(\sigma_0)$ and $G_-(\sigma_0)$ meet and separates them into two disjoint branches (§ 5, case ii). This induces a new system of waves for $\sigma_0 \leqslant 2\Omega$ and splits $\mathscr{G}_2$ into two for higher frequencies.

It is interesting to note that the waves propagating along the z axis with lengths $2\pi/(2\Omega \pm \sigma_0)$ are absent with stratification. Also when $\sigma_0 = N$, whatever may be the rotation, the disturbance propagates in all directions.

2. Equations of motion and formal solution

Consider an unbounded inviscid incompressible non-diffusive liquid rotating like a rigid body about the vertical axis OZ with a constant angular velocity Ω and with an undisturbed mean density distribution $\bar{\rho}(z)$ increasing with depth. The equations governing the motion of the liquid referred to a frame $OXYZ$ rotating with the liquid, are

$$\rho\, d\mathbf{q}/dt + 2\Omega\rho\hat{k}\times\mathbf{q} + \nabla p + \hat{k}g\rho = 0, \tag{1}$$

$$\nabla\cdot\mathbf{q} = 0, \quad d\rho/dt = 0, \tag{2), (3}$$

where $\mathbf{q} = (u, v, w)$, p, ρ and g denote the velocity vector, pressure, density and acceleration due to gravity respectively and $\hat{k}$ is the unit vector along OZ. Assuming that the motions are small and linearizing about a state of rest in the frame $OXYZ$, (1)–(3) become

$$\partial\mathbf{q}/\partial t + 2\Omega\hat{k}\times\mathbf{q} + (1/\bar{\rho})\nabla p + g\hat{k}(\rho'/\bar{\rho}) = 0, \tag{4}$$

$$\nabla\cdot\mathbf{q} = 0, \tag{5}$$

$$\partial\rho'/\partial t + \mathbf{q}\cdot\nabla\bar{\rho} = 0, \tag{6}$$

and now $\mathbf{q}$, p, ρ' denote the perturbed quantities over the undisturbed state. Eliminating p, ρ', u and v from (4)–(6), we get

$$\left[\frac{\partial^2}{\partial t^2}\nabla^2 + N^2\nabla_h^2 + 4\Omega^2\frac{\partial^2}{\partial z^2} - \frac{N^2}{g}\left(\frac{\partial^2}{\partial t^2} + 4\Omega^2\right)\frac{\partial}{\partial z}\right] w = 0, \tag{7}$$

in which $N = (-(g/\bar{\rho})\,d\bar{\rho}/dz)^{\frac{1}{2}}$ is the Brunt–Väisälä frequency (assumed to be constant), ∇^2 and ∇_h^2 are the three-dimensional and horizontal Laplacians. If we invoke Boussinesq approximation in (4), that is if $\bar{\rho}$ is replaced by $\rho_0(=\bar{\rho}(0))$ in the inertial terms, in place of (7) we get

$$\left[\frac{\partial^2}{\partial t^2}\nabla^2 + N^2\nabla_h^2 + 4\Omega^2\frac{\partial^2}{\partial z^2}\right] w = 0. \tag{8}$$

In this paper we are concerned only with (8). When w is determined from (8) the other components of the velocity can be found from (6) and the vertical component of the vorticity equation, viz.

$$\partial\omega_3/\partial t = 2\Omega\,\partial w/\partial z.$$

For periodic disturbances with frequency σ, (8) is hyperbolic or elliptic according as σ lies inside or outside the interval $(N, 2\Omega)$ or $(2\Omega, N)$. $N > 2\Omega$ corresponds to the case of a strongly stratified rotating liquid in which the buoyancy forces are dominant over the Coriolis forces while $N < 2\Omega$ is the opposite case. The former situation arises in the regions of thermoclines in the sea and the latter may arise in well-mixed regions.

Following the technique of Lighthill (1967), the nature and propagation of the waves generated by a forcing region of finite extent have been studied. If the forcing region (it could be anything like an oscillatory source or a moving body, etc.) moves along the z axis with a uniform velocity U and oscillates with a frequency σ_0 it will produce waves of frequency $\sigma_0 + Un$ (due to Doppler effect) where $\mathbf{k} = (l, m, n)$ is the wave-number vector. The effect of the forcing region can be incorporated in the governing differential equation (8) by replacing the right-hand side by a non-zero forcing term

$$e^{-i\sigma_0 t} f(\mathbf{r} - \mathbf{U}t), \tag{9}$$

where $\mathbf{r} = (x, y, z)$ is the position vector.

The differential equation (8) admits a plane-wave solution of the type

$$w = w_0 \exp\left[i(-\sigma t + lx + my + nz)\right], \tag{10}$$

if the dispersion relation

$$S(\sigma_0) \equiv \left[(\sigma_0 + Un)^2 - N^2\right](l^2 + m^2) + \left[(\sigma_0 + Un)^2 - 4\Omega^2\right] n^2 = 0 \tag{11}$$

is satisfied (since $\sigma = \sigma_0 + Un$). If $f(\mathbf{r})$ vanishes outside a finite region, taking Fourier transforms a formal solution of (8) with (9) as its right-hand side may be obtained as

$$w = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F(\mathbf{k}) \exp\{i[-\sigma_0 t + \mathbf{k}.(\mathbf{r} - \mathbf{U}t)]\}}{S(\sigma_0)}\, dl\, dm\, dn, \tag{12}$$

where $$f(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{r}}\, dl\, dm\, dn.$$

A method of obtaining a unique solution (satisfying the radiation condition) of the integral (12) is explained in Lighthill (1960, 1967). It is shown that the amplitude of the waves generated by the forcing term (9) is asymptotically

$$\frac{4\pi^2}{|\kappa|^{\frac{1}{2}} R} \frac{F(k)}{\nabla S(\sigma_0)}, \tag{13}$$

(provided $\kappa \neq 0$) where $R = |\mathbf{r} - \mathbf{U}t|$, ∇ is the gradient operator in (l, m, n) space and κ is the Gaussian curvature of the surface (11). The amplitude of the waves corresponding to a monoclastic curve of (11) decays like $R^{-\frac{5}{6}}$ instead of R^{-1}.

The directional distribution of waves, their surfaces of constant phase and the effect of N, 2Ω and σ_0 on them are discussed in the following sections.

3. The wave-number surface and the surfaces of constant phase

The equation (11) represents a surface of revolution, denoted by $S(\sigma_0)$, in the wave-number space, the n axis being the axis of revolution. It consists of two disjoint branches, which we denote by $S_\pm(\sigma_0)$, meeting the n axis at

$$A_\pm(\sigma_0) = -(\sigma_0 \mp 2\Omega)/U$$

and are bounded by the asymptotes $B_\pm(\sigma_0) \equiv n - (\sigma_0 \mp 2\Omega)/U = 0$ (see figures 1 and 3). (It must be noted that the notation $S_\pm(\sigma_0)$ denotes $S_+(\sigma_0)$ and $S_-(\sigma_0)$ and not $S_+(\sigma_0)$ or $S_-(\sigma_0)$.) $S(\sigma_0)$ also depends critically on $f^2 = 4\Omega^2/N^2$ and is curved outwards or inwards to the region bounded by $B_\pm(\sigma_0)$ according as $f \gtrless 1$ and it coincides with $B_\pm(\sigma_0)$ when $f = 1$. Besides σ_0, $A_\pm(\sigma_0)$ depend only on Ω and $B_\pm(\sigma_0)$ depend on N. When $\Omega \to 0$, $A_+(\sigma_0)$ and $A_-(\sigma_0)$ move closer and coincide with $(0, 0, -\sigma_0/U)$. Similarly $B_+(\sigma_0)$ and $B_-(\sigma_0)$ coincide with $n = -\sigma_0/U$ when $N \to 0$.

The monoclastic curves on $S(\sigma_0)$ are the circles generated by the points of inflexion of the meridional sections of the surface $S(\sigma_0)$ and they are the real roots, with $|\eta|$ lying between N and 2Ω, of the equation

$$\eta^5 - 3\sigma_0\eta^4 + 2N^2\eta^3 + 8\Omega^2\sigma_0\eta^2 - 12\Omega^2N^2\eta + 4\Omega^2N^2\sigma_0 = 0, \tag{14}$$

where $\eta = \sigma_0 + Un$. There is always one m.c. on $S_-(\sigma_0)$ corresponding to the negative real root of (14) and $S_+(\sigma_0)$ has a m.c. whenever σ_0 does not lie between N and 2Ω.

The shape of the wave crests and the surfaces of constant phase, which are the polar reciprocals of $S(\sigma_0)$, are given by the parametric equation (Lighthill 1967, p. 730)

$$\rho = (x^2+y^2)^{\frac{1}{2}} = A\left|\frac{U(4\Omega^2-\eta^2)^{\frac{1}{2}}(\eta^2-N^2)^{\frac{3}{2}}}{\eta(4\Omega^2-N^2)(\eta-\sigma_0)^2}\right|,$$

$$z = A\left|\frac{U(4\Omega^2-\eta^2)(\eta^2-N^2)}{\eta(4\Omega^2-N^2)(\eta-\sigma_0)^2} - \frac{U}{\eta-\sigma_0}\right| \operatorname{sgn}\left(\frac{\eta-\sigma_0}{U}\right), \tag{15}$$

where $A^{\frac{1}{2}}$ is a constant value of the phase $\mathbf{k}\,.\,\mathbf{r}$ and $N \leqslant |\eta| \leqslant 2\Omega$ (or $2\Omega \leqslant |\eta| \leqslant N$). The equation (15) represents two surfaces $F_\pm(\sigma_0)$ corresponding to positive and negative values of η and they are the polar reciprocals of $S_{\pm}(\sigma_0)$ respectively (see figures 2 and 4). The surfaces (15) will have a cuspidal edge corresponding to a monoclastic curve on $S(\sigma_0)$ (Lighthill 1960, p. 410). $F_{-}(\sigma_0)$ is always cusp shaped with a circular edge of regression which we denote by R_{-}. It is tangential to the z axis at the point T_{-}, the inverse point of the point at infinity on $S_{-}(\sigma_0)$ and cuts the z axis orthogonally at the point Q_{-} which is the inverse point of $A_{-}(\sigma_0)$. Whenever there is a m.c. on $S_{+}(\sigma_0)$ its polar reciprocal $F_{+}(\sigma_0)$ has similar features and the corresponding points are denoted by R_{+}, T_{+} and Q_{+}.

The phase velocity of the waves relative to the undisturbed fluid is given by $\mathbf{V}_p = \sigma\mathbf{k}/|\mathbf{k}|^2$ or

$$\frac{\mathbf{V}_p}{U} = \left|\frac{\eta(\eta^2-N^2)^{\frac{1}{2}}}{(\eta-\sigma_0)(4\Omega^2-N^2)^{\frac{1}{2}}}\right| \tag{16}$$

in the direction of $\eta\mathbf{k}$, where $\eta = \sigma_0 + Un$. For any particular η lying between N and 2Ω, $\mathbf{V}_p$ will be the phase velocity of the waves at the point (ρ, z) on the surface of constant phase (15). The direction and the relative magnitude of $\mathbf{V}_p$ are shown by the arrows in figures 2 and 4. There is a zero phase velocity at the points where the surface of constant phase touches the z axis. These points, therefore, remain fixed while the rest of the surface of constant phase increases (or decreases) in size remaining geometrically similar. Due to the Doppler effect the waves with maximum (or minimum) phase velocity are not always those with maximum (or minimum) wavelengths but they are given by the roots of the equation

$$\eta^3 - 2\sigma_0\eta^2 + \sigma_0^2N^2 = 0. \tag{17}$$

4. Waves excited by steady travelling forcing effects

For steady disturbances ($\sigma_0 = 0$) the wave-number surface $S(\sigma_0)$ becomes

$$l^2 + m^2 = \left(\frac{4\Omega^2 - U^2n^2}{U^2n^2 - N^2}\right)n^2. \tag{18}$$

The points of inflexion of (18) may be determined from (14) and they are given by

$$n = \pm(1/U)[-N^2 + (N^4 + 12\Omega^2N^2)^{\frac{1}{2}}]^{\frac{1}{2}} \tag{19}$$

and correspondingly there is a m.c. on each branch of $S(0)$. The arrows† of $S(0)$ point in the downstream direction as shown in figures 1 and 3 and those on m.c. make the maximum (constant) angle ϕ with the negative n axis where

$$\cot\phi = \frac{4[1+f^2-(1+3f^2)^{\frac{1}{2}}]^{\frac{1}{2}}}{[(1+3f^2)^{\frac{1}{2}}-2]^{\frac{3}{2}}}, \quad f^2 = \frac{4\Omega^2}{N^2}. \tag{20}$$

This implies that all the waves generated at the forcing region propagate only downstream confining themselves to the cone

$$C(0)\colon x^2 + y^2 - \tan^2\phi(z-Ut)^2 = 0, \quad z - Ut < 0 \tag{21}$$

† The arrows are the normals to the wave-number surface $S(\sigma_0)$ drawn in the direction of $S(\sigma_0+\delta)$, where δ is a small positive quantity. In other words, the normals are drawn in the direction of σ_0 increasing which is the direction of the group velocity. Hence the arrow at any point $\mathbf{k}$ on $S(\sigma_0)$ indicates the direction of propagation of waves whose length is $2\pi/|\mathbf{k}|$.

and the decay of the amplitude is given by (13). The surfaces of constant phase $F_{\pm}(0)$ are coincident and cusp-shaped and $C(0)$ is the locus of the cuspal points. The waves corresponding to the points on the m.c. propagate along the generators of $C(0)$ and those corresponding to the other points of $S(0)$ propagate within the cone. As the surface $S(0)$ is symmetrical about the origin, for every point $\mathbf{k}$ on $S(0) - \mathbf{k}$ is also a point on $S(0)$ but the solution which satisfies the radiation condition, viz. (13), takes only one out of each pair of points $\pm\mathbf{k}$ (Lighthill 1960, p. 407). Therefore there is only one system of waves whose lengths vary from 0 to $\pi U/\Omega$. The waves with maximum length, $\pi U/\Omega$, travel along the negative z axis. Along each direction within $C(0)$ there are two waves, a shorter one and a longer one (in fact, this will be the situation whenever the surface of constant phase is cusp-shaped) and a single wave along $C(0)$. As the direction of waves changes from z axis to $C(0)$, the lengths of shorter waves increase and the longer ones decrease.

Since $\mathbf{k}$ and $-\mathbf{k}$ correspond to the same point on the surface of constant phase $F(0)$, at each point on $F(0)$ there are two phase velocities, according as the point is considered as the inverse of $\mathbf{k}$ or $-\mathbf{k}$, which are equal in magnitude but opposite in direction. One can overcome this ambiguity by picking up the correct waves which correspond only to one in the pair $\pm\mathbf{k}$ with the help of the radiation condition which tells that only points with $(\mathbf{r}\,.\,\nabla S)/(\partial S/\partial\sigma) \leqslant 0$ contribute for the asymptotic formula (13) (see Lighthill 1960, equation (69) with ω replaced by $-\sigma$). Using that $\mathbf{r} = A\nabla S/\mathbf{k}\,.\,\nabla S$ on a surface of constant phase, in the above inequality, we get that only points with $n < 0$ contribute for (13). Hence the radiation condition eliminates completely the waves corresponding to $S_+(0)$ and the disturbance contains only those corresponding to $S_-(0)$. The relative magnitude and direction of the phase velocity, given by (16), on $F(0)$ are shown in figures 2 and 4. The phase velocities on $R_-Q_-R_-$, the portion within the cuspidal edge, are directed downstream for all f but on T_-R_-, the cone-like portion, they are directed towards or away from the axis according as $f \lessgtr 1$. The vertex T_- has a zero phase velocity and so remains fixed while Q_-, the point where $F(0)$ cuts the z axis, has the maximum velocity U.

When $f > 1$, the cuspidal edge of the wave crest is towards the vertex of $C(0)$ [see $F(0)$, figure 2]. As f increases from 1 to ∞ (i.e. as $N \to 0$), the semi-vertical angle ϕ increases from 0 to $\frac{1}{2}\pi$ and correspondingly the surface of constant phase also becomes bigger. When $f \to \infty$, the asymptotic planes $B_{\pm}(0)$ and the m.c.s move closer and finally coincide with $n = 0$ so that the portion within and outside the m.c.s respectively become a sphere and two coincident planes (as in the homogeneous case discussed by Lighthill (1967, p. 744)). So, as density decreases, the lengths of all waves increase. In particular, the lengths of the waves belonging to the portion within the m.c.s increase to attain a uniform value $\pi U/\Omega$ and when $N = 0$ these waves reduce to waves of uniform lengths with hemispherical crests and the waves corresponding to the rest of $S(0)$ reduce to waves propagating in a 'Taylor column'. Thus in a slightly stratified rotating liquid ($f > 1$) the buoyancy forces compress the downstream waves to a conical region by converting them to cusp-shaped ones and eliminating completely the unattenuated waves propagating ahead in a 'Taylor column'. Here all the waves depend on z also.

When $f = 1$, $S_{\pm}(0)$ coincide with $B_{\pm}(0)$ and the corresponding waves trail behind as a straight steady unattenuated disturbance.

For $f < 1$, the wave crests are reversed, in the sense the regression edge is away from the vertex of $C(0)$ [see $F(0)$, figure 4]. As $f \to 0$ (i.e. as $\Omega \to 0$) ϕ increases from 0 to $\frac{1}{2}\pi$. Also $A_{\pm}(0)$ and the monoclastic circles move closer and for $\Omega = 0$ they coincide with the origin. As a consequence, the lengths of all the waves, in particular those corresponding to the surface within these circles, increase and the curved portion within the cuspidal edge tends to infinity and when $\Omega = 0$ the wave-crests take the pattern of a 'flared skirt' [see Lighthill 1967, p. 743]. Thus the effect of Coriolis forces in a strongly stratified rotating liquid is to eliminate longer waves which are well away from the path of the disturbance and to fold back the rest of the waves to propagate in a cone with cusp-shaped crests.

5. Waves excited by travelling oscillatory forcing effects

We now consider the more general case of waves generated by a periodic forcing effect, with frequency σ_0, moving vertically upwards with a velocity U along the axis of rotation. When $\sigma_0 \neq 0$ the wave-number surface $S(\sigma_0)$ becomes asymmetric [see figures 1 and 3]. The nature of $S_-(\sigma_0)$ remains the same as in the steady case except that it shifts away from the origin and becomes more and more flat. The other branch $S_+(\sigma_0)$ undergoes different changes for different ranges of σ_0 in

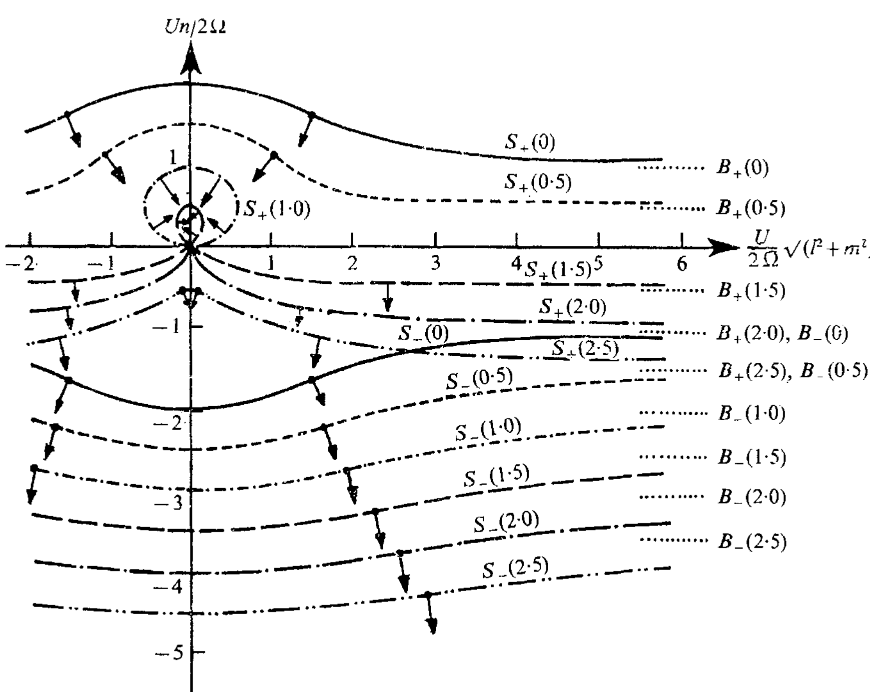

FIGURE 1. The meridional section of the wave-number surface $S(\sigma_0/2\Omega)$ for $f^2 = 4$. $B_{\pm}(\ldots)$ are the asymptotes and • denotes the point of inflexion.

relation to N and 2Ω. As a result, the two apparently coincident systems of waves corresponding to $S_{\pm}(0)$ split up.

The waves corresponding to $S_{-}(\sigma_0)$ for all f and σ_0 propagate downstream as

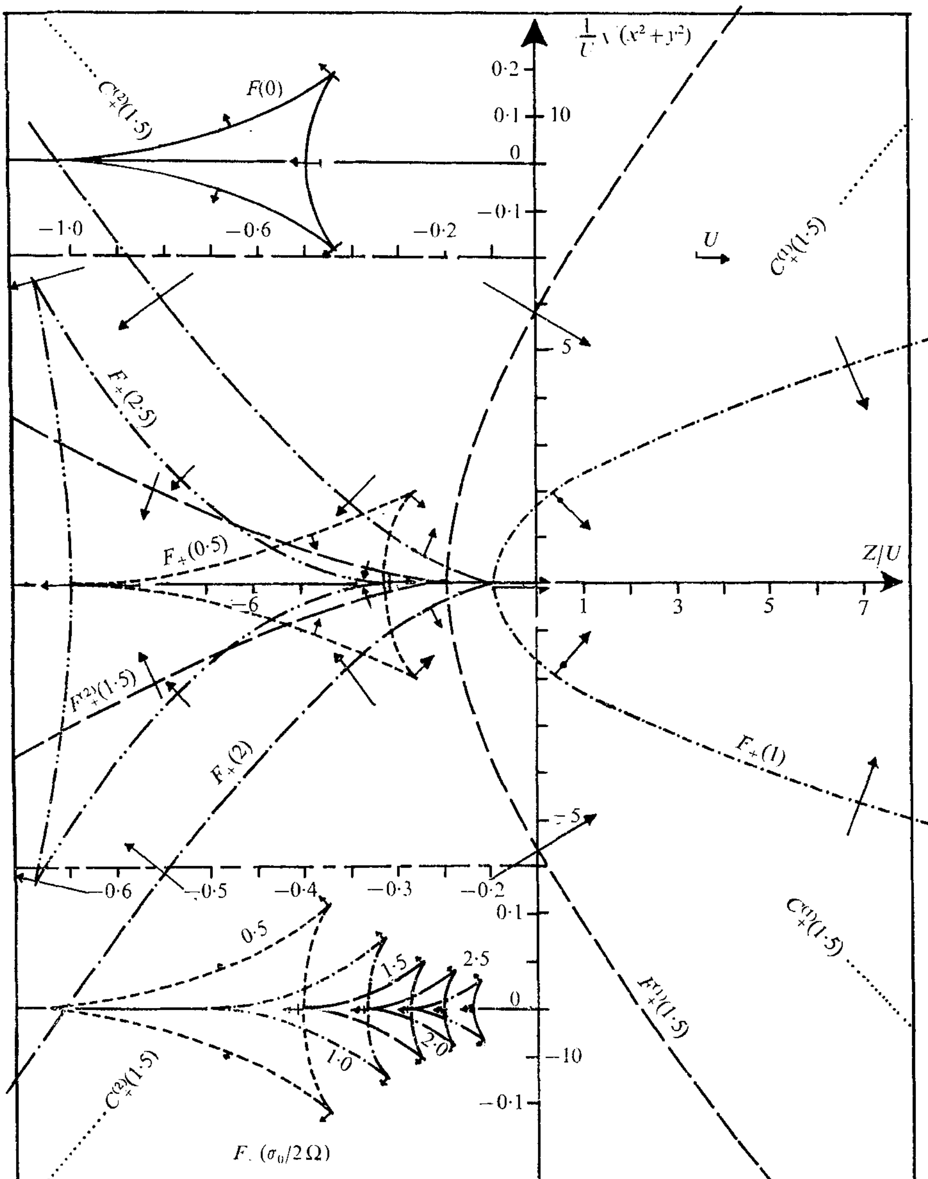

FIGURE 2. The meridional sections of the surface of constant phase corresponding to $S(\sigma_0/2\Omega)$ shown in figure 1. The inserted figures at the left-hand bottom and top corners are respectively the enlargement of the surfaces of constant phase $F_{-}(\sigma_0/2\Omega)$ and $F(0)$ corresponding to $S_{-}(\sigma_0/2\Omega)$ and $S(0)$. The main figure corresponds to those of $S_{+}(\sigma_0/2\Omega)$. All the curves are drawn for the same phase value. The arrows on the curves represent the direction and relative magnitude of the phase velocity compared with the velocity of the forcing effect U, also represented by an arrow. ─┼→ and ─●→ respectively denote the maximum and minimum phase velocities. ....., the asymptotes.

in the steady case, with cusp-shaped crests, and are confined to a cone $C_-(\sigma_0)$, determined by the m.c., and its semi-vertical angle decreases from that of $C(0)$ as σ_0 increases; the wavelengths are also decreasing. The surfaces of constant phase $F_-(\sigma_0)$ for $2\Omega > N$ and $2\Omega < N$ are shown in figures 2 and 4 and, as in the steady case, they are reversed. $F_-(\sigma_0)$ being the polar reciprocal of $S_-(\sigma_0)$, determination of the phase velocities is a simple matter and they are shown in figures 2

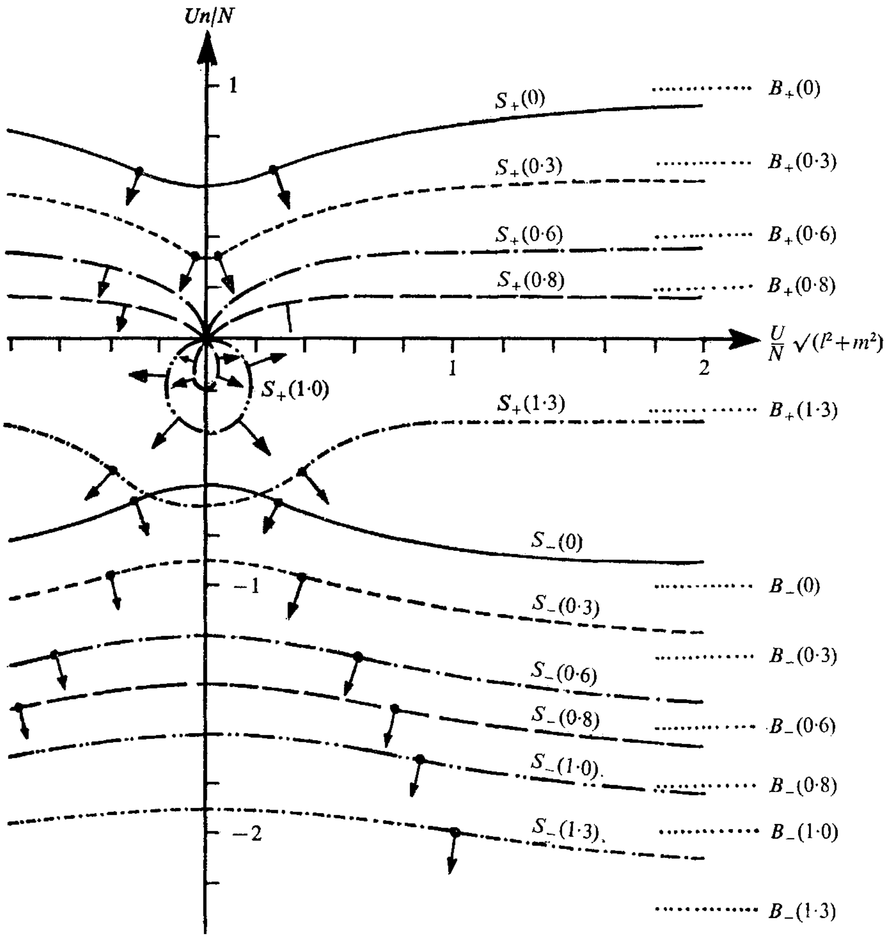

FIGURE 3. The meridional sections of the wave-number surface $S(\sigma_0/N)$ for $f^2 = 0{\cdot}36$ and for various values of σ_0/N. $B_\pm(\ldots)$ are the asymptotes and • denotes the point of inflexion.

and 4. For all f the portion within the cuspidal edge, viz. $R_-Q_-R_-$, travels in the downstream direction while the cone-like portion generated by T_-R_- travels towards or away from the z axis for $f \lessgtr 1$ respectively. For any σ_0 and for $f \geqslant 1/\sqrt{2}$ the waves with maximum phase velocity $2\Omega U/(2\Omega+\sigma)$ (which is less than U) are found along the z axis (at Q_-). But for $0 < f < 1/\sqrt{2}$ the maximum velocity appears at Q_- only when $\sigma_0 < 8\Omega^3/(N^2-8\Omega^2)$ and for higher frequencies it appears at the points which are away from the axes of rotation as shown in figure 4. They correspond to the negative root of (17). This system of waves may appear clearly for low frequencies and at higher frequencies $C_{-}(\sigma_0)$ becomes very small. In the following, we discuss the wave systems corresponding to $S_{+}(\sigma_0)$.

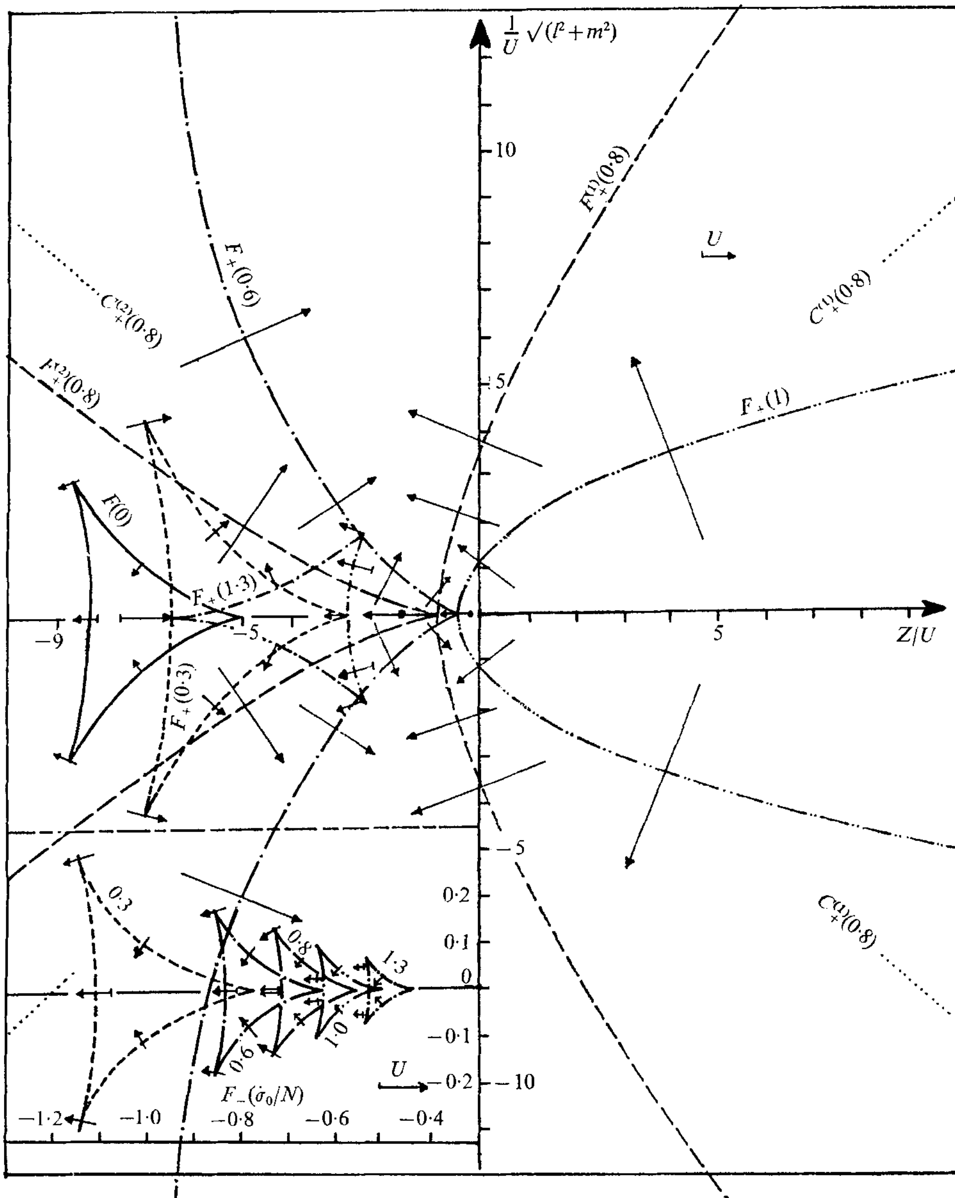

FIGURE 4. The meridional sections of the surface of constant phase corresponding to $S(\sigma_0/N)$ shown in figure 3. The main figure gives the surfaces of constant phase corresponding to $S_{+}(\sigma_0/N)$ and their phases for $\sigma_0/N = 0, 0.3, \ldots, 1.3$ are respectively in the ratio 25 : 10 : 1 : 1 : 1 : 10. The inserted figure at the left-hand bottom corner is an enlargement of the surface of constant phase corresponding to $S_{-}(\sigma_0/N)$ and they have the same phase value. The arrows have the same meaning as in figure 2.

Case (i). $N < 2\Omega$

For any σ_0, $S_+(\sigma_0)$ and $S_+(0)$ have similar characteristics, namely $S_+(\sigma_0)$ lies entirely to the positive side of the plane $n = 0$ and it has a m.c. [see $S_+(0{\cdot}5)$, figure 1]. For frequencies much less than N the arrrows of $S_+(\sigma_0)$ point to the downstream direction indicating that the waves trail behind the disturbance with cusp-shaped crests [see $F_+(0{\cdot}5)$, figure 2] confining themselves to a cone $C_+(\sigma_0)$† determined by the m.c., viz. the cone with vertex at the forcing effect and with generators parallel to the arrows on the m.c. As σ_0 approaches N the m.c. moves closer to the asymptotic plane, its radius gradually decreases to zero and the inclination of the arrows at the m.c. increases to π. As a consequence the semi-vertical angle of $C_+(\sigma_0)$ increases to π from its value at $\sigma_0 = 0$. When σ_0 is close to N the waves propagate both upsteam and downstream except in the region outside $C_+(\sigma_0)$. As $C_+(\sigma_0)$ expands, the surface of constant phase becomes larger in such a way that R_+ goes to infinity along the cone while T_+ tends to minus infinity along the z axis. Due to the translation of $S_+(\sigma_0)$ the wavelength increases with σ_0. Unlike $F_-(\sigma_0)$, the portion within the cuspidal edge of $F_+(\sigma_0)$ travels upstream while the cone like portion shrinks towards the axis for all f. When $N^2/2\Omega < \sigma_0 < N$ the waves of maximum length $2\pi NU[(4\Omega^2-N^2)(N^2-\sigma_0^2)]^{-\frac{1}{2}}$ are found in the direction making an angle $\tan^{-1}[-N(N^2-\sigma_0^2)^{\frac{1}{2}}(4\Omega^2\sigma_0^2-N^4)^{-\frac{1}{2}}]$ with the positive z axis but for other frequencies they are always found along the z axis. From the nature of the Sturm's functions of (17) it can be seen that when $1 < f < 1{\cdot}618$ and for $1/f > \sigma_1 > (2-1/f^2)^{-1}$, where $\sigma_1 = \sigma_0/2\Omega$ the maximum V_p occurs at the smallest positive root of (17), which corresponds to a point away from the z axis. But when $\sqrt{\frac{3}{2}} < f < 1{\cdot}618$ and for $\sqrt{(27/32)}\,1/f < \sigma_1 \leqslant (2-1/f^2)^{-1}$, before attaining the maximum value, V_p reaches a local minimum at the biggest positive root of (17) and so for these frequencies the maximum V_p need not be greater than its value at Q_+. The same thing may be expected when $f > 1{\cdot}618$ since there is a maximum and also a minimum for $\sqrt{(27/32)}\,1/f < \sigma_1 < 1/f$. For $\sigma_1 \leqslant \sqrt{(27/32)}\,1/f$ [or $(2-f^{-2})^{-1}$ if $f < \sqrt{(3/2)}$] the maximum $V_p(= 2\Omega U/(2\Omega-\sigma_0))$, appears at Q_+ and it is always greater than U. In figure 2, f is taken equal to 2 and the maximum V_p, therefore, occurs at Q_+.

When $\sigma_0 = N$ the asymptote disappears and $S_+(\sigma_0)$ becomes a finite closed surface passing through the origin [see $S_+(1)$, figure 1] with arrows pointing in all directions and so the waves propagate in all directions. The waves of minimum length, $\pi U/\Omega$, propagate in the direction of the negative z axis and the wavelength increases to infinity as the azimuthal angle increases to π. The surface of constant phase $F_+(1)$ is shown in figure 2. The phase velocities on $F_+(N)$ are always directed inwards and at any point V_p is greater than U. So, as time progresses, $F_+(N)$ can overtake the forcing effect and shrinks closer to the z axis. When $f < 1{\cdot}618$, the waves at the point where $F_+(N)$ cuts the z axis has the minimum phase velocity $2\Omega U/(2\Omega-\sigma_0)$. But for $f > 1{\cdot}618$ the phase velocity

† The various cones mentioned in the discussion are not shown in figures 2 and 4, but their generators are obtained by the following rule: if the surface of constant phase has an asymptote, then the asymptote itself is a generator and for the cusp-shaped one, the line joining the cusp and the origin is a generator.

decreases from $2\Omega U/(2\Omega-\sigma_0)$ to a still smaller value, given by $\eta = 1{\cdot}618N$ and then increases indefinitely (see $F_+(1)$, figure 2).

When $N < \sigma_0 < 2\Omega$ the meridional section of $S_+(\sigma_0)$ is a nodal curve asymptotic to the line $n = (N-\sigma_0)/U < 0$ with origin as the double point [see $S_+(1{\cdot}5)$, figure 2]. At the origin the slope of the arrows to $S_+(\sigma_0)$ has a discontinuity even though the slope of $S_+(\sigma_0)$ is continuous. So the direction of the arrows clearly show that there are two systems of waves; the first one corresponding to the closed portion of the surface $S_+(\sigma_0)$ and the other to the remaining portion. The first system of waves propagates both upstream and downstream except in the cone $C_+^{(1)}(\sigma_0)$ given by

$$x^2+y^2-(z-Ut)^2\cot^2\psi = 0, \quad z-Ut > 0 \tag{22}$$

and the latter inside the cone $C_+^{(2)}(\sigma_0)$, given by (22) with $z-Ut < 0$, where

$$\psi = \tan^{-1}[(4\Omega^2-\sigma_0^2)/(\sigma_0^2-N^2)]^{\frac{1}{2}} \tag{23}$$

is the angle made by the arrow to the closed portion of $S_+(\sigma_0)$ at the origin with the negative n axis. The shapes of crests are given by $F_+^{(1)}(1{\cdot}5)$ and $F_+^{(2)}(1{\cdot}5)$ in figure 2 but it must be noted that $F_+^{(1)}(\sigma_0)$ and $F_+^{(2)}(\sigma_0)$ will not meet each other for all σ_0. As σ_0 increases to 2Ω the node on $S_+(\sigma_0)$ shrinks to a cusp at the origin with the n axis as the common tangent and ψ increases to $\frac{1}{2}\pi$. So as σ_0 increases the cones $C_+^{(1)}(\sigma_0)$ and $C_+^{(2)}(\sigma_0)$ become bigger and $F_+^{(1)}(\sigma_0)$ becomes more and more feeble and moves away from the origin while $F_+^{(2)}(\sigma_0)$ moves towards the origin. The nature of the phase velocities on $F_+^{(1)}(\sigma_0)$ is similar to $F_+(N)$ but it travels with greater speed. The point at which $F_+^{(1)}(\sigma_0)$ cuts the z axis always has the minimum phase velocity $2\Omega U/(2\Omega-\sigma_0)$. $F_+^{(2)}(\sigma_0)$ always lies below the horizontal plane passing through the point at which $F_+^{(2)}(\sigma_0)$ touches the z axis and it travels towards the z axis with speed increasing with depth.

When $\sigma_0 = 2\Omega$, $S_+(2)$ in figure 1 shows that the first system of waves corresponding to the closed portion disappears completely and there will be only one system of waves propagating in all directions below the forcing region. The shape of the crests is given by $F_+(2)$ in figure 2.

When $\sigma_0 > 2\Omega$, $S_+(\sigma_0)$ lies entirely in the region $n < 0$ and a m.c. re-appears on it. The corresponding waves are therefore cusp-shaped and propagate downstream within the cone $C_+(\sigma_0)$ determined by the m.c., but now the cuspidal edge is away from the vertex of $C_+(\sigma_0)$ [see $F_+(2{\cdot}5)$, figure 2]. As σ_0 increases, $C_+(\sigma_0)$ becomes smaller and the wavelengths increase. Here the $\mathbf{V}_p$ on $R_+Q_+R_+$ is directed downstream while on T_+R_+ is directed towards the z axis and the maximum $V_p (= 2\Omega U/(\sigma_0-2\Omega))$ occurs at Q_+ and the maximum $V_p \gtreqless U$ according as $\sigma_0 \lesseqgtr 4\Omega$.

Case (ii). $2\Omega < N$

The meridional sections of $S_+(\sigma_0)$ for various values of σ_0 and $f^2 = 0{\cdot}36$ are shown in figure 3. A close comparison of figures 1 and 3 shows that even though the propagation of crests is different, the qualitative features of the waves, viz. the propagation of waves and the shape of the wave crests, in the present case are analogous to what we have discussed in case (i) and are pointed out below. For

$2\Omega < \sigma_0 < N$ the meridional section of $S_+(\sigma_0)$ has the shape of a nodal curve but now the closed portion lies in the negative side with arrows pointing outwards and the infinite branch lies in the positive side with arrows pointing in the downstream direction. The direction of the arrows and the shape of the surface of constant phase (see $S_+(0{\cdot}8)$ in figure 3 and $F_+^{(1)}(0{\cdot}8)$ and $F_+^{(2)}(0{\cdot}8)$ in figure 4) show that there are two systems of waves propagating exactly as in case (i). For $\sigma_0 = 2\Omega$ and $\sigma_0 = N$ the qualitative nature of the waves and the shape of the surface of constant phase is similar to case (i) [see $S_+(0{\cdot}6)$, $S_+(1)$, $F_+(0{\cdot}6)$ and $F_+(1)$ in figures 3 and 4]. The waves and the corresponding cones, in both the cases, undergo the same changes when σ_0 changes from N to 2Ω. For σ_0 outside the interval $[2\Omega, N]$, $S_+(\sigma_0)$ has a m.c. and the propagation of waves for $\sigma_0 < 2\Omega$ and $\sigma_0 > N$ are respectively similar to those in case (i) for $\sigma_0 > 2\Omega$ and $\sigma_0 < N$. As in case (i), the semi-vertical angle of $C_+(\sigma_0)$ tends to $\frac{1}{2}\pi$ and π respectively when σ_0 tends to 2Ω and N. But the propagation of waves crests in case (ii) and in its analogous situation of case (i) are opposite to one another, that is to say, the phase velocities in one case are reverse of the other. Whenever $1 \leqslant \sigma_0/N \leqslant f^3/(2f^2-1)$ (or ∞ if $f < 1/\surd 2$) the maximum phase velocity occurs at points which are away from the z axis and for all other frequencies the maximum (or minimum if $2\Omega \leqslant \sigma_0 \leqslant N$) appears only at the points where the surface of constant phase cuts the z axis.

Thus for all N and 2Ω there are two systems of waves for frequencies outside $[N,\ 2\Omega]$ or $[2\Omega, N]$ propagating downstream with cusped crests and three systems otherwise.

We will next consider the influence of N and 2Ω for a fixed σ_0. If the plane wave solutions were to exist then (11) restricts the Doppler-shift frequency, $|\sigma_0 + Un|$, to the values between N and 2Ω, thereby confining $S(\sigma_0)$ to the region between $A_\pm(\sigma_0)$ and $B_\pm(\sigma_0)$. Wavelengths corresponding to $S_-(\sigma_0)$ and $S_+(\sigma_0)$ (for $N < 2\Omega < \sigma_0$ or $\sigma_0 < 2\Omega < N$) are respectively bounded above by $2\pi U/(2\Omega + \sigma_0)$ and $2\pi U/(2\Omega - \sigma_0)$, which are the lengths of waves propagating along the negative z axis, while those corresponding to the closed portion of $S_+(\sigma_0)$ for $N < \sigma_0 < 2\Omega$ or $2\Omega < \sigma_0 < N$ are bounded below by $2\pi U/(2\Omega - \sigma_0)$. But the maximum length of the waves corresponding to $S_+(\sigma_0)$ for $N^2/2\Omega < \sigma_0 < N < 2\Omega$ or $2\Omega < N < \sigma_0 < \frac{1}{2}N^2$ depends both on N and 2Ω. As N or Ω (or both) increases, due to stretching out of $A_\pm(\sigma_0)$ or $B_\pm(\sigma_0)$ (or both), $S(\sigma_0)$ becomes bigger and so the waves propagating in any particular direction become shorter. The semi-vertical angles of all the cones depend on f and they all have a maximum value when $N = 0$ or $\Omega = 0$ and decrease gradually as $f \to 1$. When $f = 1$, $S_\pm(\sigma_0)$ coincide with the planes $B_\pm(\sigma_0)$ and therefore the created disturbance consists of oscillatory unattenuated waves propagating parallel to the z axis in the downstream direction.

In the homogeneous case ($N = 0$) the asymptotic planes $B_\pm(\sigma_0)$ coalesce into a single plane $n = -\sigma_0/U$ (see Nigam & Nigam 1962) and the two branches of the wave-number surface meet at infinity. So the surfaces of constant phase of both the systems $\mathscr{P}_2$ and $\mathscr{P}_3$ (see introduction), which correspond to the wave-number surface on $n < 0$, touch the axis of rotation at the point $(0, 0, -AU/\sigma_0)$. If stratification is also present, then $B_\pm(\sigma_0)$ are separated by a distance $2N/U$ and hence the two corresponding points, the points where $F_-(\sigma_0)$ and $F_+(\sigma_0)$ (or $F_+^{(2)}(\sigma_0)$ if $N < \sigma_0 < 2\Omega$) touch the z axis, also get separated by a distance

$$AU\left[\frac{1}{|N-\sigma_0|}-\frac{1}{|N+\sigma_0|}\right].$$

The nature and propagation of waves for $2\Omega > N$ and $\sigma_0 > N$ basically have the same features as those in a homogeneous rotating liquid. But the waves corresponding to $S_+(\sigma_0)$ for $\sigma_0 < N < 2\Omega$ have no counterparts in the homogeneous case and they are induced by the buoyancy forces. The presence of stratification splits the wave crests, decreases the lengths and shrinks the enveloping cones.

Just as stratification splits the wave crests in a homogeneous rotating liquid, rotation also has a similar effect on waves in a stratified liquid. A comparison of $S(\sigma_0)$ [in case (ii)] and the corresponding wave-number surface for $\Omega = 0$ (viz.

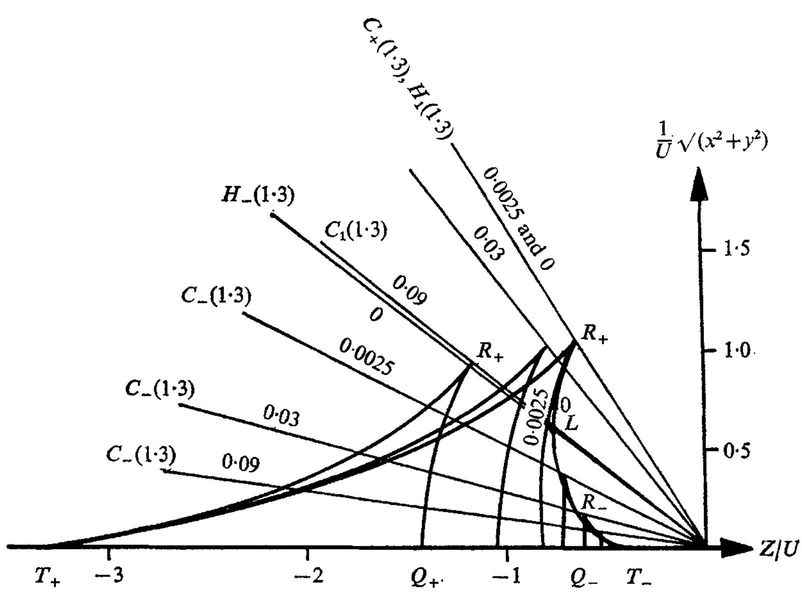

FIGURE 5. The figure shows how the meridional sections of $F_-(\sigma_0/N)$ for $f < 1$ and $\sigma_0/N = 1\cdot3$ coalesce into a single crest $T_+R_+T_-$ as $f \to 0$. The values on the curves denote the value of f. $T_+R_+S_+$ and $T_-R_-S_-$ are the meridional sections of $F_+(\sigma_0/N)$ and $F_-(\sigma_0/N)$ and when $f = 0$ they go into $T_+R_+L_+$ and LS_-. ——, the generator of the cone mentioned against it.

$P(\omega, \alpha, \beta)$ in figure 1 of Stevenson 1969) shows that when rotation is introduced, $G_1(\sigma_0)$ and $G_-(\sigma_0)$ split at $(0, 0, -\sigma_0/U)$ and separate into two disjoint branches $S_+(\sigma_0)$ and $S_-(\sigma_0)$, which are separated by a distance $2\Omega/U$. Therefore the single system of waves corresponding to $G_-(\sigma_0)$ and $G_1(\sigma_0)$ propagating within $H_1(\sigma_0)$ (or outside the extension of $H_+(\sigma_0)$ if $\sigma_0 < N$) splits into two systems of waves; the first one propagating in $C_-(\sigma_0)$ and the second one propagating in $C_+(\sigma_0)$ or outside $C_+(\sigma_0)$ according as $\sigma_0 > N$ or $2\Omega < \sigma_0 < N$. On introduction of rotation, however small it might be, $G_-(\sigma_0)$ goes into $S_-(\sigma_0)$, that is a m.c. appears on it and it cuts the n axis orthogonally. So the waves propagating in $H_-(\sigma_0)$ (here there is only one wave in each of the possible directions) fold back along the direction of the arrows at the m.c. and propagate in all directions in the smaller cone $C_-(\sigma_0)$ with cusp-shaped crests. Similarly, when $G_1(\sigma_0)$ (plus $G_+(\sigma_0)$) goes into $S_+(\sigma_0)$, the corresponding waves spread out to propagate in all directions outside $C_+^{(1)}(\sigma_0)$ if $2\Omega < \sigma_0 < N$ or inside $C_+(\sigma_0)$ if $\sigma_0 > N$, that is, they propagate in $H_-(\sigma_0)$ also. This arises from the fact that the slope at $A_+(\sigma_0)$ changes to $\frac{1}{2}\pi$ from θ_2. In figure 7, we have shown for $2\Omega < N < \sigma_0$ how the waves corresponding to $S_+(\sigma_0)$ coalesce into the waves corresponding to $G_-(\sigma_0)$ and $G_1(\sigma_0)$. Thus the splitting of crests in a strongly stratified rotating liquid for $\sigma_0 > 2\Omega$ is an effect produced by rotation. The waves appearing for $\sigma_0 < 2\Omega$ whose existence depends on Ω, have no counterparts when $\Omega = 0$ and hence their appearance may be attributed to the interaction between Coriolis and buoyancy forces.

6. Concluding remarks

If a disturbance moves vertically downwards with a uniform velocity U, the wave-number surface (11), after changing U to $-U$ becomes

$$l^2+m^2 = \frac{[4\Omega^2-(\sigma_0-Un)^2]\,n^2}{[(\sigma_0-Un)^2-N^2]}, \tag{24}$$

and this is exactly the same as (11) when n is changed to $-n$. Hence as σ_0 increases, the wave-number surface translates in the direction of positive n with arrows being reversed and now the parts played by $S_+(\sigma_0)$ and $S_-(\sigma_0)$ will be interchanged. Therefore the pattern and propagation of waves, with respect to the forcing effect, are identical in both cases. From figures 1–4 we see that as U increases $S(\sigma_0)$ is magnified uniformly in all directions while $F_\pm(\sigma_0)$ are diminished.

In the special case, when $U = 0$, (11) reduces to a double cone

$$l^2+m^2-\left[\frac{4\Omega^2-\sigma_0^2}{\sigma_0^2-N^2}\right]n^2 = 0 \tag{25}$$

and its semi-vertical angle decreases (or increases, if $N > 2\Omega$) as σ_0 increases. Hence for $N \leqslant \sigma_0 \leqslant 2\Omega$ (or $2\Omega \leqslant \sigma_0 \leqslant N$) all the waves propagate along the cone

$$x^2+y^2-[(\sigma_0^2-N^2)/(4\Omega^2-\sigma_0^2)]\,z^2 = 0, \tag{26}$$

with amplitudes decaying like $r^{-\frac{1}{2}}$ and no waves propagate within or without this cone. For σ_0 outside $[N, 2\Omega]$ the flow is similar to potential flow. The phenomenon of the 'Taylor-column' which occurs both in rotating and stratified liquids for $\sigma_0 = 0$ will now respectively appear for the values N and 2Ω of σ_0.

Let us now consider the consequences of making Boussinesq approximation. The differential equation (7) has a plane-wave solution of the type

$$\exp[\lambda z+i(lx+my+nz-\sigma t)], \quad \lambda = N^2/2g, \tag{27}$$

provided

$$[(\sigma_0+Un)^2-N^2](l^2+m^2)+[(\sigma_0+Un)^2-4\Omega^2](n^2+\lambda^2) = 0 \tag{28}$$

is satisfied. When $\lambda = 0$, (28) is identical with (11). The meridional section of (28) may be derived from those of (11) by increasing the square of the transverse wave-number, l^2+m^2, of each point by

$$[4\Omega^2-(\sigma_0+Un)^2]\,\lambda^2/[(\sigma_0+Un)^2-N^2]. \tag{29}$$

But λ being very small (it is of order 10^{-5} for an ocean) (29) is also very small unless (σ_0+Un) is very close to N. Thus Boussinesq approximation contracts the wave-number surface in the transverse direction without altering $A_{\pm}(\sigma_0)$ and the asymptotes. So the effect of Boussinesq approximation on all the waves, except those corresponding to $S_+(\sigma_0)$ for $N<\sigma_0<2\Omega$ or $2\Omega<\sigma_0<N$, is to increase their length and the semi-vertical angle of the cones by a small quantity which is almost negligible. For $N<\sigma_0<2\Omega$ and $\lambda \neq 0$, $S_+(\sigma_0)$ does not pass through the origin but cuts the $n=0$ plane in the circle

$$l^2+m^2 = \lambda^2(4\Omega^2-\sigma_0^2)/(\sigma_0^2-N^2)$$

and so the sharp corner at the origin disappears. Now, if we move along the meridional section of $S_+(\sigma_0)$ from $A_+(\sigma_0)$ to $B_+(\sigma_0)$, all the arrows lie on the right-hand side and their inclination with the n axis varies continuously. Also a m.c. appears and is close to $n=0$. So the corresponding waves propagate in a cone determined by the m.c. But (29) being very small, the surface (28) is almost coincident with $S_+(\sigma_0)$ and so the wave propagation is practically the same as in the case $\lambda=0$. Thus for frequencies in this range the Boussinesq approximation splits the waves into two systems of waves by allowing longer waves with lengths greater than $2\pi[(\sigma_0^2-N^2)/\lambda^2(4\Omega^2-N^2)]^{\frac{1}{2}}$. When $\lambda \neq 0$, $S_+(N)$ is no longer a closed surface passing through the origin, for when n is sufficiently small the term (29) is dominant and tends to infinity as $n \to 0$. Therefore, the corresponding waves, instead of propagating in all directions, propagate without penetrating into a conical region upstream. Also, there are unattenuated waves, corresponding to the portion tending to infinity along $n=0$, propagating downstream close to the n axis and are independent of z. When $\sigma_0 = 2\Omega$ the cusp of $S_+(2\Omega)$ at the origin disappears and it touches the $n=0$ plane at the origin and a m.c., which is almost coincident with the origin, appears. As a result, the corresponding waves propagate downstream in a cone with semi-vertical angle slightly less than $\frac{1}{2}\pi$ with cusped crests. But the portion within the m.c. being very small and very close to the origin, the cuspidal edge (and the inside portion) will be far away from the origin and will be almost invisible.

Finally, when the forcing effect is two-dimensional i.e. $m\equiv 0$, it is easy to see that the waves propagate exactly as in the axisymmetric case, with the exception that now the waves generated are two-dimensional in character, the cones become wedges and the amplitude decays like $R^{-\frac{1}{2}}$.

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